RANDOM WALKS IN SLIGHTLY CHANGING ENVIRONMENTS

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Chapter 1

Introduction

1.1 Overview

Consider any simple, undirected, connected, and locally-finite[1] graph $G = (V, E)$. There are several random processes on G that are of interest, but perhaps the most classical and fundamental process is the *simple random walk* (SRW) on G. Namely, the SRW on G is the stochastic process $\{X_t\}_{t=0}^{\infty}$ such that for any $t \in \mathbb{N} := \{0, 1, 2, \dots\}$, we have $X_t \in V$ and

$$\mathbb{P}(X_{t+1} = y \mid X_t = x) = \frac{1}{\deg(x)}$$

for all $\{x, y\} \in E$. In words, at each timestep the process transitions to any neighboring vertex with equal probability. Of course, the distribution of X_0 (or the initial distribution) may be arbitrary.

Since G is connected and locally-finite, V is countable, and let us assume that V is infinite. Then, it is natural to ask whether it is possible for the SRW to get "lost" in the graph G. This is captured by notions of *recurrence* and *transience*, which are basic questions regarding SRWs.

Definition 1 (Recurrence and Transience)**.** Consider the return probability

$$p = \mathbb{P}(\text{There is } T > 0 \text{ such that } X_T = X_0).$$

Then, $\{X_t\}_{t=0}^{\infty}$ is *recurrent* if $p = 1$ and *transient* if $p < 1$.

[1] This means that every vertex has finite degree.

Since $\{X_t\}_{t=0}^{\infty}$ is an irreducible, time-homogeneous Markov chain, one can easily check that $\{X_t\}_{t=0}^{\infty}$ is either recurrent for any initial distribution or transient for any initial distribution. Hence, it is well-defined to say that G is recurrent (resp. transient) if $\{X_t\}_{t=0}^{\infty}$ is recurrent (resp. transient). Moreover, since the transition probabilities are fixed over time, we see that recurrence (resp. transience) implies that a.s. X_0 is visited infinitely (resp. finitely) often. In fact, since G is connected, it follows that a.s. every $v \in V$ is visited infinitely (resp. finitely) often: For any $x, y \in V$, the probability of the process eventually visiting y from x is at least some positive constant p_{xy} which is independent of time.

The pioneering result in this area that comes with an initial surprise is Pólya's recurrence theorem [15], which states that the SRW on $\mathbb{Z}^d$ is recurrent if $d \in \{1, 2\}$ and transient if $d \geq 3$. An elementary proof can be given by counting the number of paths that return to the origin after $2n$ steps. Indeed, this relies on the fact that $\mathbb{Z}^d$ is bipartite and space-homogeneous. One may wonder if there is a general theory to determine whether G is recurrent or transient: It turns out there is a nice connection with electrical network theory that becomes more natural once we slightly generalize our setup. We discuss the generalization below.

Equip G with any weight function $w : E \to (0, \infty)$. Then, one can consider the random walk $\{X_t\}_{t=0}^{\infty}$ on the weighted graph (G, w), where transition probabilities are now proportional to edge-weights. Namely, for any $t \in \mathbb{N}$, we have

$$\mathbb{P}(X_{t+1} = y \mid X_t = x) = \frac{w(x, y)}{\sum_{y \sim x} w(x, y)}$$

for all $y \sim x$ (which means $\{x, y\} \in E$). Since $\{X_t\}_{t=0}^{\infty}$ is still a time-homogeneous Markov chain on V, all discussions above regarding recurrence or transience also hold in the weighted case. In particular, it is again well-defined to say that (G, w) is recurrent or transient. Roughly speaking, the connection with electrical network theory begins by viewing (G, w) as an electrical network, where each $e \in E$ is a resistor that has conductance $w(e)$. We will further review this connection in section 1.2 below.

The main mathematical object of this paper is a further generalization of weighted random walks, where we remove the assumption of time-homogeneity. For instance, one can imagine how the conductance of an actual resistor could physically change over time. We make this generalization by assuming that at each time $t \in \mathbb{N}$, the weight function on

G is given by a random variable w_t. Such processes are generally known as *random walks in changing environments* (RWCE). Again, the basic question of interest is recurrence or transience, but the various implications that were true in the time-homogeneous case may no longer coincide for general RWCEs. Hence, one needs more care in making the question concrete. We will formally discuss RWCEs in section 1.3.

A complete theory for recurrence or transience of RWCEs is far out of reach compared to the time-homogeneous case. Current main questions revolve around finding natural, general conditions on $\{w_t\}_{t=0}^{\infty}$ that guarantee recurrence or transience of the RWCE. In chapter 2, we will illustrate some techniques that are used to show the recurrence or transience of RWCEs. In section 2.1, we focus on monotone bounded RWCEs, which include open problems that are currently of most interest [2]. Here, the main proof idea is to construct a super/submartingale involving voltages on the electrical network G and then use the optional stopping theorem to bound the probability of return. In section 2.2, we consider a non-monotone RWCE on $\mathbb{Z}$ with uniformly converging weights. In this special case, we show that the Lyapunov central limit theorem is enough to prove a phase transition previously observed by [14] which depends on the rate of convergence of the weights.

To conclude, in chapter 3, we present our main result which is also the novel contribution of this thesis. Namely, we explore the limits of the martingale approach from section 2.1 and derive a condition for recurrence or transience that holds for any graph G. This can be viewed as an attempt to overcome the fact that the voltage sequence, which is a natural martingale on a single weighted graph, is no longer as nice on a sequence of weighted graphs (or in changing environments).

1.2 Electrical Network Theory

In this section, we recall important results on the characterization of recurrence or transience given by electrical network theory. The first chapter of [10] provides a nice introduction with detailed proofs for the interested reader. Further standard references include [1, 9, 13].

We begin with some notation. Fix any $G = (V, E)$ with the usual assumptions and take any $w : E \to (0, \infty)$. Fix any $s \in V$, which we consider as the origin of G. For $n \geq 0$, let $V_n := \{v \in V : d(s, v) \leq n\}$ and $\partial V_n := \{v \in V : d(s, v) = n\}$ where d is the shortest-path distance on G. Finally, let $\{X_t\}_{t=0}^{\infty}$ denote the weighted random walk on (G, w) and τ_n denote the first time $t \in \mathbb{N}$ such that $X_t \in \partial V_n$.

Then, the return probability from (and to) $s \in V$, say p, satisfies

$$\sum_{u \sim s} \frac{w(s,u)}{w(s)} \mathbb{P}_u(\tau_n < \tau_0) \to 1 - p$$

as $n \to \infty$ where $w(s) := \sum_{u \sim s} w(s,u)$ and $\mathbb{P}_u$ is the probability measure assuming $X_0 = u$. Note that $\mathbb{P}_u(\tau_n < \tau_0)$ only depends on the finite subgraph of G induced by V_n, which we denote as $G_n = (V_n, E_n)$. The key observation of electrical network theory is the fact that $\mathbb{P}_u(\tau_n < \tau_0)$ equals the voltage of u, say $v(u)$, when viewing (G_n, w) as an electrical network with s as the source and ∂V_n as the sink so that $v(x) = 1$ for any $x \in \partial V_n$. This follows from the fact that both $v(\cdot)$ and $\mathbb{P}_\cdot(\tau_n < \tau_0)$ are *harmonic* on $V_n \setminus (\{s\} \cup \partial V_n)$ with equal boundary conditions (see [10] for further details).

We now elaborate further on the basic theory of finite electrical networks. For this part alone, let $G = (V, E)$ be finite with weights $w : E \to (0, \infty)$. Fix $s \in V$ as the *source* and $t \in V \setminus \{s\}$ as the *sink*. The central objects of electrical network theory are s/t flows which are defined below.

Definition 2 (s/t flows)**.** *We say that $i : V^2 \to [0, \infty)$ is an s/t flow if*

$$\begin{aligned} i(u,v) &= -i(v,u), \\ i(u,v) &= 0 \text{ if } \{u,v\} \notin E, \\ J_u := \sum_{v \sim u} i(u,v) &= 0 \text{ for any } u \notin \{s,t\}. \end{aligned}$$

One can check that $J_s = -J_t$, and we define $|i| := |J_s|$ as the *size* of the flow i. We say that i is a *unit flow* if $|i| = 1$. Moreover, we assign an *energy* to the flow i, given by

$$E(i) := \sum_{e \in E} i^2(e) r(e)$$

where $r(e) := 1/w(e)$ is the *resistance* of $e \in E$. Similarly, $w(e)$ can be thought as the conductance of edge e.

Among all possible **unit** s/t flows on (G, w), there is a unique flow i which obtains the minimum possible energy. We call i the *Kirchoff flow*, as i is exactly the unique flow that satisfies the Kirchoff potential law: Namely, we have

$$\sum_{j=1}^{n} i(v_j, v_{j+1}) r(v_j, v_{j+1}) = 0$$

for any cycle $v_1, v_2, \ldots, v_n, v_{n+1} = v_1$. The Kirchoff flow is nice as it induces a voltage function $v : V \to [0, \infty)$ where $v(s) = 0$ and

$$v(u) := \sum_{j=1}^{n} i(v_j, v_{j+1}) r(v_j, v_{j+1})$$

for any path $(s = v_1, v_2, \ldots, v_{n+1} = u)$ from s to u. We remark that this is only well-defined when i is a Kirchoff flow.

The energy of the Kirchoff flow i also gets a special name, and we say that

$$\mathcal{R}(s, t) := \sum_{e \in E} i^2(e) r(e)$$

is the *effective resistance* between s and t in (G, w). As mentioned above, the effective resistance equals the minimum possible energy among all unit flows, and this is known as Thomson's principle. Intuitively, the network (G, w) can be reduced to a single resistor between s and t with resistance $\mathcal{R}(s, t)$. Also, we remark that the voltage at t induced by i will exactly equal $\mathcal{R}(s, t)$. Hence, to get $v(t) = 1$, one should rescale the flow i so that $|i| = 1/\mathcal{R}(s, t)$.

We are now ready to describe the connection between recurrence or transience and electrical network theory. Let $\mathcal{R}_n$ denote the effective resistance between s and ∂V_n in (G_n, w). Indeed, ∂V_n may not be a single vertex, and in this case we identify all vertices in ∂V_n into a single vertex while removing any self-loops that occur. An extremely useful property of effective resistance is given by Rayleigh's monotonicity principle, which states that the effective resistance of a network can only increase if we only increase edge-resistances. It follows that $\{\mathcal{R}_n\}_{n=0}^{\infty}$ is increasing, hence the limit $\mathcal{R} := \lim_{n\to\infty} \mathcal{R}_n$ always exists. The amazing result of the theory determines the return probability p as

$$p = 1 - \frac{1}{w(s)\mathcal{R}}.$$

In particular, the random walk on (G, w) is recurrent if and only if $\mathcal{R} = \infty$.

1.3 Random Walk in Changing Environments

Here, we formally introduce our main object of study. Let $(\Omega, \mathcal{F}, \mathbb{P})$ denote the underlying probability space.

Definition 3 (RWCE)**.** A *random walk in changing environment* (RWCE) on a graph $G = (V, E)$ is a stochastic process $\{\langle X_t, w_t \rangle\}_{t \in \mathbb{N}}$ such that for any $y \in V$, we have

$$\mathbb{P}(X_{t+1} = y \mid \mathcal{F}_t) = \frac{w_t(X_t, y)}{\sum_{z \sim X_t} w_t(X_t, z)}$$

where $X_t : \Omega \to V$, $w_t : \Omega \to (0, \infty)^E$, and $\mathcal{F}_t = \sigma(X_0, \ldots, X_t, w_0, \ldots, w_t)$ for each $t \in \mathbb{N}$.

In words, at time $t \in \mathbb{N}$, the RWCE traverses a neighboring edge from X_t with probability proportional to its weight, which is given by the realization of w_t. By requiring the edge-weights to be positive, we are only considering *proper* RWCEs. Moreover, for any $\{x, y\} \notin E$, we write $w_t(x, y) = 0$ by convention.

While the term RWCE was coined by Amir et al. in [2], it includes the large class of self-interacting walks which were studied even before. A well-known example is the linearly edge-reinforced random walk by Coppersmith and Diaconis [5] from the eighties. Other examples include the once-reinforced random walk [6] or the “true” self-avoiding walk with bond repulsion [16].

As aforementioned in our discussion in section 1.1, we need to be more careful when discussing recurrence or transience of general RWCEs. This is because the return probability may change over time, and it is also not obvious whether infinite visits to a single vertex implies infinite visits to every vertex. We give a simple example below that shows how different time-inhomogeneous processes can be compared to time-homogeneous ones.

Example 1. Let $G = (V, E)$ where $V = \{a, b, c\}$ and $E = \{\{a, b\}, \{b, c\}\}$. Let $X_0 = a$ and take $\{w_t\}_{t=0}^{\infty}$ where $w_t(a, b) = 1$ and $w_t(b, c) = (t + 1)^2$ for each $t \geq 0$. Finally, let $\{\langle X_t, w_t \rangle\}_{t \in \mathbb{N}}$ denote the resulting RWCE on G. Of course, each w_t is actually deterministic in this case.

For each $t \geq 0$, consider the event $A_t = \{X_{2t+2} = a\}$. Since $X_{2t+1} = b$ holds for any

$t \geq 0$, we see that

$$\sum_{t=0}^{\infty} \mathbb{P}(A_t) = \sum_{t=0}^{\infty} \frac{1}{1+(t+1)^2} < \infty.$$

Thus, by the first Borel-Cantelli lemma, vertex a is a.s. visited finitely often. However, vertex b and c are a.s. visited infinitely often. Indeed, this never occurs in the time-homogeneous case.

In most interesting examples of RWCEs that we consider, however, the various notions of recurrence and transience will coincide. Hence, we follow [2] and adopt the strongest definitions of recurrence or transience as below.

Definition 4 (Recurrence/Transience/Mixed-Type)**.** An RWCE on $G = (V, E)$ is *recurrent* if a.s. every vertex is visited infinitely often. It is *transient* if a.s. every vertex is visited finitely often. Otherwise, the RWCE is of *mixed-type*.

Of course, the RWCE in example 1 is of mixed-type. In this example, note that the probability of jumping from b to a tends to 0 as $t \to \infty$. The following condition prevents this from happening.

Definition 5 (Elliptic RWCE)**.** Let $\{\langle X_t, w_t\rangle\}_{t\in\mathbb{N}}$ be an RWCE on $G = (V, E)$. For each $\{x, y\} \in E$, assume that $\mathbb{P}(X_{t+1} = y \mid X_t = x)$, whenever well-defined, is bounded away from 0 as t varies. Then, we say that $\{\langle X_t, w_t\rangle\}_{t=0}^{\infty}$ is *elliptic* (uniformly in time).

This is useful since any elliptic RWCE that a.s. visits some vertex infinitely (resp. finitely) often is also recurrent (resp. transient). The argument is the same as the time-homogeneous case. Namely, fix any $x, y \in V$. Then, by ellipticity (and connectivity of G), whenever the RWCE is at x, the probability of eventually visiting y is at least some constant $p_{xy} > 0$ independent of time. Hence, if x is visited infinitely often, then a.s. y is visited infinitely often. The contrapositive implies that if y is visited finitely often, then a.s. x is visited finitely often.

Throughout this paper we will mostly focus on *bounded* RWCEs, which are a special case of elliptic RWCEs. We note that there are many interesting examples and problems even with this assumption.

Definition 6 (Bounded RWCE)**.** Let $\{\langle X_t, w_t\rangle\}_{t\in\mathbb{N}}$ be an RWCE on $G = (V, E)$. Then, the RWCE is *bounded* if there are deterministic weights $a, b : E \to (0, \infty)$ so that a.s. $a(e) \leq w_t(e) \leq b(e)$ for each $t \in \mathbb{N}$ and $e \in E$.

Finally, a particularly important question is whether there is an essential difference between *adaptive* and *nonadaptive* RWCEs regarding recurrence or transience.

Definition 7 (Adaptive/Nonadaptive)**.** An RWCE is *nonadaptive* if the distribution of w_{t+1} given $w_0, \ldots w_t$ is independent of $X_0, \ldots, X_t$. Otherwise, the RWCE is adaptive.

We will illustrate this question further in section 2.1 below.

Chapter 2

Two Illustrative Problems

The purpose of this chapter is to introduce interesting problems and techniques involving recurrence or transience of bounded RWCEs. We begin with the monotone bounded problem, then show a phase transition for a non-monotone RWCE on $\mathbb{Z}$.

2.1 Monotone Bounded RWCEs

Let $G = (V, E)$ be a graph under the usual conditions and $\{\langle X_t, w_t\rangle\}_{t\in\mathbb{N}}$ be an RWCE on G. Consider any condition on $\{w_t\}_{t\in\mathbb{N}}$ that requires (G, w_t) to be a.s. recurrent for each $t \in \mathbb{N}$. In order to demonstrate a fundamental difference between adaptive and nonadaptive RWCEs, we are most interested in conditions that imply recurrence for nonadaptive RWCEs but not for adaptive RWCEs. Of course, one can develop the same question for the transient case, which we will also consider. For sake of simplicity, however, we will focus on the recurrent case in the following discussions.

An interesting candidate for such a condition was given by Amir et al. in [2]. Namely, they required $\{w_t\}_{t\in\mathbb{N}}$ to be a.s. increasing (edgewise) and bounded above by some recurrent (G, w). This means $w_t(e) \le w_{t+1}(e) \le w(e)$ for each $t \in \mathbb{N}$ and $e \in E$. By Rayleigh's monotonicity principle, it follows that each (G, w_t) is a.s. recurrent. In [2], they further constructed an adaptive RWCE on $\mathbb{Z}^2$ that satisfies the monotone-bounded condition but is transient. However, it remains an open question to show that any nonadaptive walk satisfying the monotone-bounded condition must be recurrent. We term this problem as the *monotone bounded problem* for RWCEs. Also, we note that the analogous case for transience (decreasing weights bounded below by a transient graph) was partially affirmed by Dembo et al. in [8].

An important remark is that we are interested in results for arbitrary graphs G. If G is required to be a tree, then Amir et al. [2] showed that there is no difference between adaptive and nonadaptive walks under the monotone-bounded condition: One will always have recurrence. This can be thought as a "lower-bound" result for trees, as there is no difference regarding adaptiveness under any condition implying the monotone-bounded condition for trees. However, the monotone bounded problem is completely open for general graphs G.

The situation is quite similar even when we restrict our attention to a special type of monotone bounded RWCEs, namely once-reinforced random walks. A once-reinforced random walk on G is an RWCE where $w_0(e) = 1$ for any $e \in E$ and $w_t(e) = 1 + \delta$ for some universal $\delta > 0$ if $e \in E$ has been crossed at least once up to time t. When G is a tree, Collevecchio et al. completely characterized the regions of δ that imply recurrence or transience by introducing a quantity called the branching-ruin number [4]. In contrast, on $\mathbb{Z}^2$ the question of recurrence or transience remains completely open. We remark that partial progress has been made by Kious et al. [12] for graphs of the form $\mathbb{Z} \times \Gamma$ where Γ is finite.

In the remaining section, we discuss the proof of Amir et al.'s result on the monotone bounded problem for trees which we mentioned above. The proof illustrates a standard technique of showing recurrence by constructing a supermartingale and then using the optional stopping theorem to bound the probability of return.

Proposition 1 (Amir et al.)**.** *Let $G = (V, E)$ be any tree with the usual assumptions. Let $\{\langle X_t, w_t\rangle\}_{t\in\mathbb{N}}$ be any RWCE on G such that $\{w_t\}_{t\in\mathbb{N}}$ is increasing and bounded above by (G, w_∞) which is recurrent (we assume w_0, w_∞ are deterministic). Then, $\{X_t\}_{t\in\mathbb{N}}$ is also recurrent.*

Before proceeding to the proof, we briefly recall the optional stopping theorem as it will be used throughout this paper. Let $\{X_t\}_{t\in\mathbb{N}}$ be a supermartingale (resp. submartingale) and τ be a stopping time with respect to the filtration $\{\mathcal{F}_t\}_{t\in\mathbb{N}}$. The optional stopping theorem states that if there is $C > 0$ such that $|X_{t\wedge\tau}| \le C$ for all $t \in \mathbb{N}$, then $\mathbb{E}[X_\tau] \le \mathbb{E}[X_0]$ (resp. $\mathbb{E}[X_\tau] \ge \mathbb{E}[X_0]$). Of course, there are other versions of the theorem as well, but for our purposes this version suffices. We now proceed to the proof.

Proof of Proposition 1. Fix the origin $s \in V$ and assume that $X_0 = x \in V \setminus \{s\}$ is determined. Also fix n large enough so that $x \in V_n$ and consider $G_n = (V_n, E_n)$. Let i denote the unit Kirchoff flow on (G_n, w_∞) from s to ∂V_n and define

$$f_t(v) := \sum_e i(e) r_t(e)$$

for each $v \in V_n$ where the sum is over edges of the **unique** path from s to v. First, it is clear that $\{f_t(v)\}_{t\in\mathbb{N}}$ is decreasing since $\{w_t\}_{t\in\mathbb{N}}$ is increasing. Next, by definition of a flow, one can see that

$$\begin{aligned}\mathbb{E}[f_t(X_{t+1}) \mid \mathcal{F}_t] &= f_t(X_t) + \sum_{y\sim X_t} \frac{w_t(X_t, y)}{w_t(X_t)} i(e) r_t(X_t, y) \\ &= f_t(X_t)\end{aligned}$$

if $t < \tau$ where $\tau = \inf\{t \in \mathbb{N} : X_t \in \{s\} \cup \partial V_n\}$. Hence, we see that

$$\mathbb{E}[f_{t+1}(X_{t+1}) \mid \mathcal{F}_t] \leq \mathbb{E}[f_t(X_{t+1}) \mid \mathcal{F}_t] = f_t(X_t)$$

if $t < \tau$ and thus $\{g_{t\wedge\tau}\}_{t\in\mathbb{N}}$ is a supermartingale where $g_t = f_t(X_t)$. Since f_t is decreasing over time and V_n is finite, we see that $|g_t| \leq C$ for some constant $C > 0$ independent of t. By the optional stopping theorem,

$$\mathbb{E}[f_\tau(X_\tau)] \leq \mathbb{E}[f_0(x)].$$

From our definition above, for any $y \in \partial V_n$, note that $f_\infty(y)$ is the voltage at y induced by i, hence $f_\infty(y) = \mathcal{R}_n$ which is the effective resistance between s and ∂V_n in (G_n, w_∞). Since f_t is decreasing, $f_t(y) \geq f_\infty(y) = \mathcal{R}_n$ for any $t \in \mathbb{N}$ and we get

$$\mathbb{E}[f_\tau(X_\tau)] \geq \mathcal{R}_n \cdot \mathbb{P}(X_\tau \in \partial V_n)$$

since $f_\tau(s) = 0$. Thus,

$$\mathbb{P}(X_\tau \in \partial V_n) \leq \frac{\mathbb{E}[f_0(x)]}{\mathcal{R}_n} \to 0$$

as $n \to \infty$ since (G, w_∞) is recurrent and

$$f_0(x) \le \sum_{s \to x} r_0(e)$$

which is a constant independent of n. In other words, the RWCE a.s. returns to s. At any time of return, the RWCE is again monotone bounded, hence the RWCE returns to s infinitely often. Finally, since the RWCE is bounded, a.s. every vertex is visited infinitely often and we conclude our proof. □

The key part of Amir et al.'s proof above is constructing the supermartingale $f_t(X_t)$. However, this doesn't work for general graphs G since there may be multiple paths from s to v and $f_t(v)$ may not even be well-defined. In chapter 3, we construct a different supermartingale that cannot imply a result as strong as Proposition 1 but nonetheless works for any graph G.

2.2 A Non-Monotone RWCE on $\mathbb{Z}$

In this section, we consider a nonadaptive, non-monotone RWCE on the nearest-neighbor graph of $\mathbb{Z}$, say $G = (V, E)$. Moreover, for notational convenience we index time by $t \in \{1, 2, \dots\}$. For each $t \ge 1$ and $x \in \mathbb{Z}$, let

$$w_t(x-1, x) := 1 + (-1)^{x+t} \varepsilon_t$$

where $\varepsilon_t = 1/t^\alpha$ for some constant $\alpha > 0$. Then, we obtain a sequence of deterministic weight functions $\{w_t\}_{t=1}^\infty$. In fact, w_1 is not proper, but we will allow this throughout this section. For any fixed t, note that $w_t(e)$ alternates between $1 + \varepsilon_t$ and $1 - \varepsilon_t$ as one traverses $e \in E$ from left to right. Similarly, for any fixed $e \in E$, the values of $w_t(e)$ alternate as $1 \pm \varepsilon_1, 1 \mp \varepsilon_2, 1 \pm \varepsilon_3, \dots$. In particular, we see that $\{w_t\}_{t=1}^\infty$ is non-monotone.

Let $\{\langle X_t, w_t \rangle\}_{t=1}^\infty$ denote the RWCE on G where $\{w_t\}_{t=1}^\infty$ are given as above. Also let $X_1 = 0$. In this section, we prove the following phase transition which was previously observed in [14].

Proposition 2. *Let $\{\langle X_t, w_t \rangle\}_{t=1}^\infty$ denote the RWCE above. Then, $\{X_t\}_{t=1}^\infty$ is recurrent if $\alpha \ge 1/2$ and transient if $\alpha < 1/2$.*

Let w_∞ denote the all-ones weight function on E so that $w_t \to w_\infty$ edgewise. For any $\alpha > 0$, note that $\sup_{e\in E} |w_t(e) - w_\infty(e)| = \varepsilon_t \to 0$ as $t \to \infty$. Hence, Proposition 2 shows that uniform convergence of weights is not sufficient, and the rate of convergence is in fact crucial. On the other hand, considering Proposition 1 for any once-reinforced random walk on $\mathbb{Z}$ shows that uniform convergence of weights is also not necessary.

The proof of Proposition 2 crucially relies on the fact that X_t is expressible as the sum of t independent Bernoulli variables, hence we are able to use classical tools from probability theory. We begin with the proof of the transience case of Proposition 2.

Proof of Proposition 2 (Transience). Since G is bipartite and $X_1 = 0$, note that for each $t \geq 1$,

$$X_t = \sum_{k=1}^{t-1} \xi_k$$

where the ξ_k are independent and $\xi_k = 1$ with probability $(1 + \varepsilon_k)/2$ and $\xi_k = -1$ otherwise. Moreover,

$$\mathbb{E}[X_t] = \sum_{k=1}^{t-1} \varepsilon_k.$$

Since $\xi_k \in [-1, 1]$, by Hoeffding's inequality, we have

$$\mathbb{P}(X_t = 0) \leq \mathbb{P}(|X_t - \mathbb{E}[X_t]| \geq \mathbb{E}[X_t]) \leq \exp\left(-\frac{\mathbb{E}[X_t]^2}{2t-2}\right)$$

for $t > 1$. Since $\mathbb{E}[X_t]^2/(2t - 2) > 0$, we further have

$$\mathbb{P}(X_t = 0) \leq \frac{j!}{\left(\frac{\mathbb{E}[X_t]^2}{2t-2}\right)^j}$$

for any $j \geq 0$. Since $\mathbb{E}[X_t] = \Theta(t^{1-\alpha})$, we see that $\mathbb{P}(X_t = 0) = O(1/t^{(1-2\alpha)j})$ for any $j \geq 0$. Thus, if $(1 - 2\alpha)j > 1$ or equivalently $\alpha < (1 - 1/j)/2$, then

$$\sum_{t=1}^{\infty} \mathbb{P}(X_t = 0) < \infty.$$

Indeed, by Borel-Cantelli, we a.s. have $X_t = 0$ finitely often. Moreover, since $\{w_t\}_{t=2}^\infty$ is bounded, it follows that a.s. every vertex in $\mathbb{Z}$ is visited finitely often. Letting $j \to \infty$, we conclude that $\{X_t\}_{t=1}^\infty$ is transient for all $\alpha < 1/2$ as desired. □

We now show the recurrent case of Proposition 2.

Proof of Proposition 2 (Recurrence). We will construct a deterministic sequence of times $\{t_k\}_{k=1}^{\infty}$ such that $t_k \geq 1$ and

$$\mathbb{P}(X_{t_{k+1}} < 0 \mid X_{t_k}) \geq \beta$$

for some constant $\beta > 0$ independent of k. We claim that this implies that a.s. $X_t = 0$ infinitely often. To see this, first note that if $X_t < 0$, then a.s. there is $T > t$ such that $X_T = 0$. This follows from a simple coupling argument as the SRW on $\mathbb{Z}$ is recurrent, while our RWCE always has a positive bias to the right (which diminishes over time). On the other hand, if $X_t > 0$, a.s. there must exist $t_j > t$ such that $X_{t_j} < 0$. This implies the existence of some $T \in (t, t_j)$ such that $X_T = 0$. Hence, for any t such that $X_t \neq 0$, a.s. there is $T > t$ such that $X_T = 0$, thus $X_t = 0$ infinitely often.

Moreover, since $\{w_t\}_{t=2}^{\infty}$ is bounded, it would further follow that a.s. every vertex in $\mathbb{Z}$ is also visited infinitely often. Hence, to show recurrence, it suffices to show that for any $\alpha \geq 1/2$, we can construct the sequence $\{t_k\}_{k=1}^{\infty}$ as above. For our desired property, one can easily see it is enough to show that

$$\mathbb{P}(X_{t_{k+1}} < 0 \mid X_{t_k} = t_k - 1) \geq \beta.$$

Fix any $\alpha \geq 1/2$ and let $Z \sim N(0,1)$. Let $\beta = \mathbb{P}(Z \leq -3)/2 > 0$. We will inductively construct $\{t_k\}_{k=1}^{\infty}$. Choose $t_1 = 1$. Next, assume that we have successfully determined t_k. We will show how to determine t_{k+1}. For any $t > t_k$, given $X_{t_k} = t_k - 1$, we have $X_t = t_k - 1 + S_{k,t}$ where $S_{k,t} = \xi_{t_k} + \xi_{t_k+1} + \cdots + \xi_{t-1}$. Then,

$$\mathbb{E}[S_{k,t}] = \sum_{j=t_k}^{t-1} \varepsilon_j,$$

$$\mathrm{Var}[S_{k,t}] = (t - t_k) - \sum_{j=t_k}^{t-1} \varepsilon_j^2.$$

Since t_k is a constant, note that $\mathbb{E}[S_{k,t}] \sim \frac{1}{1-\alpha} t^{1-\alpha}$ and $\mathrm{Var}[S_{k,t}] \sim t$. Since $S_{k,t}$ is a sum of independent variables and $|\xi_j - \varepsilon_j| \leq 2$ for any $j \geq 1$, one can easily see that

Lyapunov's condition is satisfied for $S_{k,t}$, namely

$$\lim_{t\to\infty} \frac{\sum_{j=t_k}^{t-1} \mathbb{E}|\xi_j - \varepsilon_j|^3}{\mathrm{Var}[S_{k,t}]^{3/2}} = 0.$$

Hence, the central limit theorem (CLT) holds for $S_{k,t}$ (in this case known as the Lyapunov CLT) and we have

$$Y_{k,t} := \frac{S_{k,t} - \mathbb{E}[S_{k,t}]}{\sqrt{\mathrm{Var}(S_{k,t})}} \to Z$$

in distribution as $t \to \infty$.

Next, note that

$$a_{k,t} := \frac{-t_k - \mathbb{E}[S_{k,t}]}{\sqrt{\mathrm{Var}(S_{k,t})}} \to c_\alpha$$

as $t \to \infty$ where $c_\alpha = -2$ if $\alpha = 1/2$ and $c_\alpha = 0$ if $\alpha > 1/2$. Thus, there exists T_1 such that $a_{k,t} \geq -3$ for all $t \geq T_1$. Hence, we have

$$\mathbb{P}(X_t < 0 \mid X_{t_k} = t_k - 1) \geq \mathbb{P}(S_{k,t} \leq -t_k) = \mathbb{P}(Y_{k,t} \leq a_{k,t}) \geq \mathbb{P}(Y_{k,t} \leq -3)$$

for all $t \geq T_1$. Moreover, since $\mathbb{P}(Y_{k,t} \leq -3) \to \mathbb{P}(Z \leq -3)$, there exists T_2 such that $\mathbb{P}(X_t < 0 \mid X_{t_k} = t_k - 1) \geq \mathbb{P}(Z \leq -3)/2$ for all $t \geq T_2$. Taking $t_{k+1} = \max\{T_2, t_k\}$ gives our desired construction and we conclude our proof. $\square$

Given $\varepsilon_t = c/t^\alpha$, we remark that the results from Proposition 2 hold for any constant $c > 0$. In general, one can construct a similar RWCE on any recurrent tree such that there is always an outward bias that diminishes over time. An interesting open question is determining the phase transition of such processes, and whether the threshold may differ among different trees. The above techniques no longer work as X_t is no longer expressible as a sum of independent Bernoulli variables.

Chapter 3

Main Result

In this chapter, we follow the martingale method discussed in sections 1.1 and 2.1 to derive a condition for recurrence or transience of bounded RWCEs on general graphs. The following is our main result.

Theorem 1. *Let $G = (V, E)$ be any graph and (G, w_0) be recurrent (resp. transient) where $w_0 \in (0, \infty)^E$ is deterministic. Let $\{\langle X_t, w_t \rangle\}_{t \in \mathbb{N}}$ be any bounded RWCE on G. If there is $C > 0$ so that*

$$\sum_{t,e} |r_t(e) - r_{t+1}(e)| \leq C$$

almost surely, then $\{X_t\}_{t \in \mathbb{N}}$ is recurrent (resp. transient).

From Rayleigh's monotonicity principle, one can easily check that if

$$\sum_{e \in E} |r_t(e) - r_{t+1}(e)| < \infty,$$

then (G, w_t) is recurrent (resp. transient) if and only if (G, w_{t+1}) is recurrent (resp. transient). Hence, under our condition, all (G, w_t) are a.s. recurrent (resp. transient). Our condition is quite restrictive as we are requiring a double-summation to be bounded. Nonetheless, the condition includes cases where $w_t \neq w_{t+1}$ for infinitely many $t \in \mathbb{N}$. Moreover, it holds for any graph G.

Before discussing the proof, we state some corollaries that follow from Theorem 1.

Corollary 1. *Let $G = (V, E)$ be any graph and (G, w_0) be recurrent (resp. transient) where $w_0 \in (0, \infty)^E$ is deterministic. Let $\{\langle X_t, w_t \rangle\}_{t \in \mathbb{N}}$ be any bounded RWCE on G*

that is also edgewise monotone. If there is $C > 0$ so that

$$\sum_{e \in E} |r_0(e) - r_\infty(e)| \leq C$$

almost surely where

$$r_\infty(e) := \lim_{t \to \infty} r_t(e),$$

then $\{X_t\}_{t \in \mathbb{N}}$ is recurrent (resp. transient).

Proof. This follows immediately from Theorem 1. The edgewise limits a.s. exist since our condition implies that $\{r_t(e)\}_{t \in \mathbb{N}}$ is Cauchy for each $e \in E$. Moreover, note that the direction of monotonicity can be different for each edge. $\square$

If we limit our interest to a deterministic sequence of weights, we can further alleviate the condition of summing over all edges.

Corollary 2. *Let $G = (V, E)$ be any graph of bounded degree. Let $\{w_t\}_{t \in \mathbb{N}}$ be a deterministic sequence of weights in $(0, \infty)^E$ so that $w_t(e) \to w(e)$ for each $e \in E$ and*

$$\sum_{t \in \mathbb{N}} \varepsilon_t < \infty$$

where $\varepsilon_t := \sup_{e \in E} |r_t(e) - r(e)|$ for each $t \in \mathbb{N}$. If (G, w) is recurrent (resp. transient), then the RWCE $\{X_t\}_{t \in \mathbb{N}}$ is also recurrent (resp. transient).

Proof. Assume that (G, w) is recurrent. We will create an adaptive process that has the same distribution as $\{X_t\}_{t \in \mathbb{N}}$. Namely, consider the RWCE $\{\langle X'_t, w'_t \rangle\}_{t \in \mathbb{N}}$ where $X'_0 = X_0$ and

$$w'_t(e) := \begin{cases} w_t(e) & e \text{ is adjacent to } X'_t, \\ w(e) & e \text{ is not adjacent to } X'_t \end{cases}$$

for each $t \in \mathbb{N}$. Since transitions are local, one can easily see that $\{X'_t\}_{t \in \mathbb{N}} = \{X_t\}_{t \in \mathbb{N}}$ in distribution. Next, assume $X_0 = s \in V$ is determined. Then, w'_0 is also deterministic, and (G, w'_0) is recurrent since it differs from (G, w) finitely. It remains to show that Theorem 1 is applicable to $\{\langle X'_t, w'_t \rangle\}_{t \in \mathbb{N}}$ where $X'_0 = s$.

First, we show that $\{w'_t\}_{t \in \mathbb{N}}$ is bounded. Fix any $e \in E$. Since $\varepsilon_t \to 0$, there is t' such that $\varepsilon_t \leq r(e)/2$ for all $t \geq t'$. Thus, $|r_t(e) - r(e)| \leq \varepsilon_t$ and $r(e)/2 \leq r_t(e) \leq 3r(e)/2$ for all $t \geq t'$. This shows that $\inf_{t\in\mathbb{N}} r_t(e) > 0$ and $\sup_{t\in\mathbb{N}} r_t(e) < \infty$ for each $e \in E$, thus $\{w_t\}_{t\in\mathbb{N}}$ is bounded. In particular, $\{w'_t\}_{t\in\mathbb{N}}$ is also bounded.

To conclude, note that for any $e \in E$, we have

$$|r'_t(e) - r'_{t+1}(e)| \leq |r'_t(e) - r(e)| + |r(e) - r'_{t+1}(e)| \leq \varepsilon_t + \varepsilon_{t+1}.$$

Moreover, r'_t and r'_{t+1} differ by at most 2Δ edges where $\Delta := \sup_{v\in V} \deg(v)$. Thus,

$$\sum_{e\in E} |r'_t(e) - r'_{t+1}(e)| \leq 2\Delta(\varepsilon_t + \varepsilon_{t+1})$$

which gives

$$\sum_{t,e} |r'_t(e) - r'_{t+1}(e)| \leq 2\Delta \sum_{t\in\mathbb{N}}(\varepsilon_t + \varepsilon_{t+1}) \leq 4\Delta \sum_{t\in\mathbb{N}} \varepsilon_t.$$

Since the right-most term is a finite constant, we conclude by Theorem 1 that $\{X'_t\}_{t\in\mathbb{N}}$ and thus $\{X_t\}_{t\in\mathbb{N}}$ is recurrent. The proof for transience is exactly the same. □

3.1 Proof Overview

Here, we give an overview of our proof of Theorem 1. Let $\{\langle X_t, w_t\rangle\}_{t\in\mathbb{N}}$ be any RWCE on $G = (V, E)$ satisfying the conditions of Theorem 1. Since $\{w_t\}_{t\in\mathbb{N}}$ is bounded, recall that our RWCE is in particular elliptic. Hence, to show that our RWCE is recurrent (resp. transient), it suffices to show that some vertex is a.s. visited infinitely (resp. finitely) often.

Our proof consists of two main parts. First, we show Theorem 1 in the special case where s has a single neighbor. To do so, we use construct a super/submartingale of the form $\{f_t(X_t)\}_{t\in\mathbb{N}}$ and use the optional stopping theorem as discussed above. In particular, the construction works on any graph as we consider the maximum/minimum ratio of vertex-voltages across a single time step. Since s has a single neighbor, for any $v \neq s$, the voltage at v is positive and thus ratios are well-defined.

Next, when s has multiple neighbors, we attach a new vertex s' to s and construct a new RWCE whose recurrence or transience implies the recurrence or transience of the original RWCE. Then, we apply the first part above to this new RWCE by considering

s' as the origin of G'. This concludes the proof.

3.2 The Single Neighbor Case

Throughout this section, we assume that the origin s has a single neighbor. We aim to show the following special case of Theorem 1.

Lemma 1. *Let $G = (V, E)$ be any graph and $w_0 \in (0, \infty)^E$ be deterministic such that (G, w_0) is recurrent (resp. transient). Assume the origin $s \in V$ has a single neighbor and let $\{\langle X_t, w_t \rangle\}_{t \in \mathbb{N}}$ be any bounded RWCE on G. If there is $C > 0$ so that*

$$\sum_{t,e} |r_t(e) - r_{t+1}(e)| \leq C$$

almost surely, then $\{\langle X_t, w_t \rangle\}_{t \in \mathbb{N}}$ is recurrent (resp. transient).

Indeed, choosing the origin is arbitrary and it suffices for a vertex of degree one to exist. As mentioned above, we will construct a super/submartingale and then apply the optional stopping theorem to derive a condition for recurrence or transience. Then, we will show that this condition is satisfied assuming the condition of Lemma 1.

3.2.1 Super/submartingales

We first construct the desired super/submartingale. Assuming that the origin s and the RWCE $\{\langle X_t, w_t \rangle\}_{t \in \mathbb{N}}$ are given, we recall some notation involving electrical networks. For $n \geq 0$, let $V_n := \{v \in V : d(s, v) \leq n\}$ and $\partial V_n := \{v \in V : d(s, v) = n\}$ where d is the shortest-path distance on G. Let $G_n = (V_n, E_n)$ denote the subgraph induced in G by V_n. For $n \geq 1, t \in \mathbb{N}$, and $u \in V_n$, let $v_{n,t}(u)$ denote the (random) voltage of u in (G_n, w_t) when s is grounded and ∂V_n is kept at voltage 1. If $u \notin V_n$, define $v_{n,t}(u) := 1$.

Again, the key connection between random walks on graphs and electrical networks is that whenever w_t is fixed, $v_{n,t}(u)$ equals the probability that the weighted random walk $\{Z_k\}_{k \in \mathbb{N}}$ on (G_n, w_t) with $Z_0 = u \in V_n$ will hit ∂V_n before s. In particular, both quantities are harmonic, meaning $\{v_{n,t}(Z_{k \wedge \theta})\}_{k \in \mathbb{N}}$ is a martingale with respect to $\{Z_k\}_{k \in \mathbb{N}}$ where $\theta := \inf\{k \in \mathbb{N} : Z_k \in \{s\} \cup \partial V_n\}$. In our case, the analogous process for $\{\langle X_t, w_t \rangle\}_{t \in \mathbb{N}}$ is $\{v_{n,t}(X_{t \wedge \tau})\}_{t \in \mathbb{N}}$ where $\tau = \inf\{t \in \mathbb{N} : X_t \in \{s\} \cup \partial V_n\}$. Unfortunately, for arbitrary

$u \in V_n$ the sequence $\{v_{n,t}(u)\}_{t\in\mathbb{N}}$ is not necessarily monotone and thus $\{v_{n,t}(X_{t\wedge\tau})\}_{t\in\mathbb{N}}$ is not a super/submartingale.

To bypass this difficulty, for each $t \in \mathbb{N}$ we consider the maximum/minimum of the ratio $v_{n,t+1}(u)/v_{n,t}(u)$ over all $u \in V_n \setminus \{s\}$. For $n \geq 1$ and $t \in \mathbb{N}$, let

$$\alpha_{n,t} := \max_{u\in V_n\setminus\{s\}} \frac{v_{n,t+1}(u)}{v_{n,t}(u)} \geq 1,$$
$$\beta_{n,t} := \min_{u\in V_n\setminus\{s\}} \frac{v_{n,t+1}(u)}{v_{n,t}(u)} \leq 1,$$

where the inequalities follow by considering $u \in \partial V_n$. Also, the quantities are well-defined (positive and finite) since s has a single neighbor which gives $v_{n,t} > 0$ on $V_n \setminus \{s\}$ for all $t \in \mathbb{N}$. Finally, recall that $\tau = \inf\{t \in \mathbb{N} : X_t \in \{s\} \cup \partial V_n\}$ and $\mathcal{F}_t = \sigma(X_0, \ldots, X_t, w_0, \ldots, w_t)$ for each $t \in \mathbb{N}$. The following is our desired super/submartingale.

Lemma 2. *Fix $n \geq 1$ and let*

$$A_t = \frac{v_{n,t}(X_t)}{\prod_{k=0}^{t-1} \alpha_{n,k}}, \quad B_t = \frac{v_{n,t}(X_t)}{\prod_{k=0}^{t-1} \beta_{n,k}}$$

for $t \in \mathbb{N}$. Then, $\{A_{t\wedge\tau}\}_{t\in\mathbb{N}}$ is a supermartingale and $\{B_{t\wedge\tau}\}_{t\in\mathbb{N}}$ is a submartingale with respect to $\{\mathcal{F}_t\}_{t\in\mathbb{N}}$.

Proof. It suffices to prove the supermartingale case as the submartingale case is identical. First, note that

$$A_{t+1} = \frac{v_{n,t+1}(X_{t+1})}{\prod_{k=0}^{t} \alpha_{n,k}} \leq \frac{v_{n,t}(X_{t+1})}{\prod_{k=0}^{t-1} \alpha_{n,k}}$$

by construction. Next, if $t < \tau$, we have $(t+1) \wedge \tau = t+1$ and thus

$$\mathbb{E}[A_{(t+1)\wedge\tau} \mid \mathcal{F}_t] \leq \mathbb{E}\left[\frac{v_{n,t}(X_{t+1})}{\prod_{k=0}^{t-1} \alpha_{n,k}} \mid \mathcal{F}_t\right] = \frac{1}{\prod_{k=0}^{t-1} \alpha_{n,k}} \mathbb{E}\left[v_{n,t}(X_{t+1}) \mid \mathcal{F}_t\right] = A_{t\wedge\tau}.$$

If $t \geq \tau$, then

$$\mathbb{E}[A_{(t+1)\wedge\tau} \mid \mathcal{F}_t] = \mathbb{E}\left[\frac{v_{n,\tau}(X_\tau)}{\prod_{k=0}^{\tau-1} \alpha_{n,k}} \mid \mathcal{F}_t\right] = \frac{v_{n,\tau}(X_\tau)}{\prod_{k=0}^{\tau-1} \alpha_{n,k}} = A_{t\wedge\tau}$$

as desired and we conclude our proof. $\square$

3.2.2 Optional Stopping Theorem

We now apply the optional stopping theorem to the super/submartingale constructed above. For the results of this section, we remark that it suffices to assume ellipticity instead of boundedness of the given RWCE. We begin with the supermartingale $\{A_{t\wedge\tau}\}_{t\in\mathbb{N}}$.

Lemma 3. *Let $\{\langle X_t, w_t\rangle\}_{t\in\mathbb{N}}$ be any elliptic RWCE on $G = (V, E)$ and assume the origin $s \in V$ has a single neighbor. For each $n \geq 1$, assume there is $a_n \in \mathbb{R}$ such that $\prod_{t=0}^{\infty} \alpha_{n,t} \leq a_n < \infty$ almost surely. If $\limsup_{n\to\infty} a_n < \infty$ and $v_{n,t}(u) \to 0$ almost surely as $n \to \infty$ for any $t \in \mathbb{N}$ and $u \in V$, then $\{\langle X_t, w_t\rangle\}_{t\in\mathbb{N}}$ is recurrent.*

Proof. By ellipticity, it suffices to show that s is a.s. visited infinitely often. Since $\alpha_{n,t} \geq 1$, note that the conditions of the lemma hold for any subprocess $\{\langle X_t, w_t\rangle\}_{t\geq t'}$ where $t' > 0$. Hence, it suffices to show that $X_t = s$ for some $t \in \mathbb{N}$ assuming $X_0 = u \neq s$. Fix some $n \geq 1$ and recall the supermartingle $\{A_{t\wedge\tau}\}_{t\in\mathbb{N}}$ from Lemma 2. Since $|A_{t\wedge\tau}| \leq 1$ for all $t \in \mathbb{N}$, the optional stopping theorem gives $\mathbb{E}[A_\tau] \leq \mathbb{E}[A_0]$. Hence,

$$\mathbb{E}[v_{n,0}(X_0)] \geq \mathbb{E}\left[\frac{v_{n,\tau}(X_\tau)}{\prod_{t=0}^{\tau-1} \alpha_{n,\tau}}\right] \geq \frac{\mathbb{P}(X_\tau \in \partial V_n)}{a_n}$$

which can be rearranged as $\mathbb{P}(X_\tau \in \partial V_n) \leq a_n \cdot v_{n,0}(u)$. Taking the limit superior on both sides of the inequality gives our desired result. □

Next, we proceed similarly with the submartingale $\{B_{t\wedge\tau}\}_{t\in\mathbb{N}}$.

Lemma 4. *Let $\{\langle X_t, w_t\rangle\}_{t\in\mathbb{N}}$ be any elliptic RWCE on $G = (V, E)$ and assume the origin $s \in V$ has a single neighbor x. For each $n \geq 1$, assume there is $b_n \in \mathbb{R}$ such that a.s. $\prod_{t=0}^{\infty} \beta_{n,t} \geq b_n > 0$. If $\liminf_{n\to\infty} b_n > 0$ and $\inf_{n,t} v_{n,t}(x) > 0$, then $\{\langle X_t, w_t\rangle\}_{t\in\mathbb{N}}$ is transient.*

Proof. By ellipticity, it suffices to show that s is a.s. visited finitely often. We will show that given $X_t = x$, assuming $\mathbb{P}(X_t = x) > 0$, the probability of never returning to s again is at least some positive constant independent of t. This suffices since whenever the process visits s, it must visit x the next step.

First consider when $X_0 = x$. Fix some $n \geq 1$ and recall the submartingale $\{B_{t\wedge\tau}\}_{t\in\mathbb{N}}$ from Lemma 2. Since $|B_{t\wedge\tau}| \leq 1/b_n < \infty$, the optional stopping theorem gives $\mathbb{E}[B_\tau] \geq$

$\mathbb{E}[B_0]$. Hence,

$$\mathbb{E}[v_{n,0}(X_0)] \leq \mathbb{E}\left[\frac{v_{n,\tau}(X_\tau)}{\prod_{t=0}^{\tau-1}\beta_{n,\tau}}\right] \leq \frac{\mathbb{P}[X_\tau \in \partial V_n]}{b_n}$$

which can be rearranged as $\mathbb{P}[X_\tau \in \partial V_n] \geq b_n \cdot v_{n,0}(x) \geq b_n \cdot \inf_{n,t} v_{n,t}(x)$. Taking the limit inferior on both sides as $n \to \infty$, we get $\mathbb{P}[\text{never return to } s \text{ again} \mid X_0 = x] \geq K$ for some $K > 0$.

When $X_t = x$, we can construct a submartingale similar to $B_{t\wedge\tau}$ by viewing X_t as the initial vertex and (G, w_t) as the initial graph. Since $\beta_{n,t} \leq 1$, the same method gives $\mathbb{P}[\text{never return to } s \text{ again} \mid X_t = x] \geq K$ as desired and we conclude our proof. □

3.2.3 Bounding Voltage-Ratios

Having Lemma 3 and 4, we want to use these results to prove Lemma 1. For this purpose, we estimate $\alpha_{n,t}$ and $\beta_{n,t}$ by deriving an upper bound for $|v_{n,t+1}(u)/v_{n,t}(u) - 1|$. We begin with the following expression for $|v_{n,t+1}(u) - v_{n,t}(u)|$.

Lemma 5. *For any $n \geq 1$, $t \in \mathbb{N}$, and $u \in V_{n-1} \setminus \{s\}$ we have*

$$v_{n,t+1}(u) - v_{n,t}(u) = \frac{1}{\mathcal{R}_{n,t}(s, \partial V_n)} \sum_{e=\{x,y\}\in E_n} (r_t(e) - r_{t+1}(e)) \cdot i^{n,t+1}_{u,\{s\}\cup\partial V_n}(x,y) \cdot i^{n,t}_{s,\partial V_n}(x,y)$$

where $\mathcal{R}_{n,t}(a,b)$ is the effective resistance between vertices a, b in (G_n, w_t). Also, $i^{n,t}_{v,S}$ is the unit current in (G_n, w_t) from v (which is grounded) to $S \subseteq V_n \setminus \{v\}$. Finally, $i^{n,t}_{v,S}(x,y)$ is the amount of the current $i^{n,t}_{v,S}$ across $\{x,y\}$ from x to y.

Proof. Note that all random variables in the claim are determined given w_t and w_{t+1}. The key idea is to represent $v_{n,t+1}(u)$ in terms of the current $i_1 := i^{n,t+1}_{u,s\cup\partial V_n}$. Namely, we claim that

$$v_{n,t+1}(u) = \sum_{y\in\partial V_n}\sum_{x\in V_n} i_1(x,y). \tag{3.1}$$

In words, the right-hand side of (3.1) is the total amount of current in i_1 that flows into ∂V_n. Recall that the probabilistic interpretation of $i_1(x,y)$ is given by the weighted random walk on (G_n, w_{t+1}) that begins at u and runs until hitting $s \cup \partial V_n$. Namely,

$i_1(x,y)$ equals the expected net number of crossings of $\{x,y\}$ in the given direction during the random walk. In particular, it is zero if $x \nsim y$. Taking $x \sim y$ as specified in the summation above, if $x \in s \cup \partial V_n$ we also have $i_1(x,y)=0$ as $\{x,y\}$ is never crossed. Otherwise, if $x \in V_{n-1} \setminus \{s\}$, we can cross $\{x,y\}$ exactly once during the random walk as it will terminate after crossing. Hence, $i_1(x,y)$ equals the probability that the random walk terminates after crossing $\{x,y\}$. It follows that the right-hand side of (3.1) is simply the probability that the weighted random walk on (G_n, w_{t+1}) beginning at u will hit ∂V_n before s. By the probabilistic interpretation of voltage, this is exactly $v_{n,t+1}(u)$.

The rest of our proof is routine algebra of flows, which we explain below. First, by Kirchhoff's current law we extend (3.1) to get

$$v_{n,t+1}(u) - v_{n,t}(u) = \sum_{y \in V_n} v_{n,t}(y) \sum_{x \in V_n} i_1(x,y).$$

As current is antisymmetric, we further obtain

$$\begin{aligned} v_{n,t+1}(u) - v_{n,t}(u) &= \frac{1}{2} \sum_{x,y \in V_n} (v_{n,t}(y) - v_{n,t}(x)) \cdot i_1(x,y) \\ &= \frac{1}{\mathcal{R}_{n,t}(s, \partial V_n)} \sum_{e=\{x,y\} \in E_n} r_t(e) \cdot i_0(x,y) \cdot i_1(x,y) \end{aligned}$$

where $i_0 := i^{n,t}_{s,\partial V_n}$ and the second equality is by Ohm's law [10].

To conclude, it suffices to show that

$$L := \sum_{e=\{x,y\} \in E_n} r_{t+1}(e) \cdot i_1(x,y) \cdot i_0(x,y) = 0.$$

We evaluate L by essentially reversing the above process. Let $\phi(x)$ denote the voltage of $x \in V_n$ induced by i_1. Then, by Ohm's law we have

$$\begin{aligned} L &= \sum_{e=\{x,y\} \in E_n} (\phi(y) - \phi(x)) \cdot i_0(x,y) \\ &= \frac{1}{2} \sum_{x,y \in V_n} (\phi(y) - \phi(x)) \cdot i_0(x,y) \\ &= \sum_{y \in V_n} \phi(y) \sum_{x \in V_n} i_0(x,y) \end{aligned}$$

where the second and third equalities follow since current is antisymmetric. By Kirchoff's current law, we can simplify further to obtain

$$L = \phi(s) \sum_{x \in V_n} i_0(x,s) + \sum_{y \in \partial V_n} \phi(y) \sum_{x \in V_n} i_0(x,y).$$

Note that $\phi(y) = \phi(s)$ for any $y \in \partial V_n$ and i_0 is a unit flow. Hence, we get $L = -\phi(s) + \phi(s) = 0$ as desired and conclude our proof. □

We now crucially use the assumption that s has a single neighbor to get the following corollary.

Corollary 3. *Assume that s has a single neighbor x. Then, for any $n \geq 1$, $t \in \mathbb{N}$, and $u \in V_n \setminus \{s\}$, we have*

$$\left| \frac{v_{n,t+1}(u)}{v_{n,t}(u)} - 1 \right| \leq w_t(s,x) \sum_{e \in E} |r_t(e) - r_{t+1}(e)|.$$

Proof. Since the right-hand side of Lemma 4 involves unit currents, taking absolute values gives

$$|v_{n,t+1}(u) - v_{n,t}(u)| \leq \frac{1}{\mathcal{R}_{n,t}(s, \partial V_n)} \sum_{e \in E} |r_t(e) - r_{t+1}(e)|.$$

Moreover, the inequality trivially holds for $u \in \partial V_n$. Finally, since s has a single neighbor x, we see that $v_{n,t}(u) \geq v_{n,t}(x) = r_t(s,x)/\mathcal{R}_{n,t}(s, \partial V_n)$. Combining the two inequalities gives our desired result. □

3.2.4 Proof of the Single Neighbor Case

We are now ready to prove Lemma 1.

Showing Recurrence

We begin with the recurrent case.

Proof of Lemma 1 (Recurrence). We aim to use Lemma 3. First, we check that for any $t \in \mathbb{N}$ and $u \in V_n$, we have $v_{n,t}(u) \to 0$ almost surely as $n \to \infty$. Let $d(s,u) = \ell$ and

$(x_0, \ldots, x_\ell)$ be a path from s to u. Then, for $n > \ell$ we have

$$v_{n,t}(u) = \frac{1}{\mathcal{R}_{n,t}(s, \partial V_n)} \sum_{k=0}^{\ell-1} i^{n,t}_{s,\partial V_n}(x_k, x_{k+1}) r_t(x_k, x_{k+1} \leq \frac{1}{\mathcal{R}_{n,t}(s, \partial V_n)} \sum_{k=0}^{\ell-1} r_t(x_k, x_{k+1}).$$

Next, by the boundedness condition there exists $C_1 > 0$ such that $\sum_e \delta_e \leq C_1$ almost surely where $\delta_e := \sum_{t=0}^{\infty} |r_t(e) - r_{t+1}(e)|$ for $e \in E$. Hence, it follows that $|r_0(e) - r_t(e)| \leq \delta_e \leq C_1$ and thus $r_t(e) \leq r_0(e) + C_1$. Since r_0 is deterministic, $\sum_{k=0}^{\ell-1} r_t(x_k, x_{k+1})$ is bounded and it suffices to show that a.s. $\mathcal{R}_{n,t}(s, \partial V_n) \to \infty$ as $n \to \infty$. Let $i := i^{n,t}_{s,\partial V_n}$. Then, by Thomson's principle, we a.s. have

$$\mathcal{R}_{n,0}(s, \partial V_n) \leq \sum_{e \in E_n} i^2(e) r_0(e) \leq \sum_{e \in E_n} i^2(e)(r_t(e) + \delta_e) \leq \mathcal{R}_{n,t}(s, \partial V_n) + C_1.$$

Since (G, w_0) is recurrent, it follows that a.s. $\mathcal{R}_{n,t}(s, \partial V_n) \to \infty$ for each $t \in \mathbb{N}$ as desired.

Next, we show that the condition on $\alpha_{n,t}$ holds. With the crucial assumption that the RWCE is bounded, there exists $C_2 > 0$ such that $w_t(s, x) \leq C_2$ where x is the unique neighbor of s. By Corollary 3, we get

$$\begin{aligned}
\prod_{t=0}^{\infty} \alpha_{n,t} &\leq \prod_{t=0}^{\infty} \left(1 + \sum_{e \in E} w_t(s, x) |r_t(e) - r_{t+1}(e)|\right) \\
&\leq \exp\left(\sum_{t,e} w_t(s, x) |r_t(e) - r_{t+1}(e)|\right) \leq e^{C_1 C_2}.
\end{aligned}$$

Therefore, we can choose $a_n = e^{C_1 C_2}$ in Lemma 2 for each $n \geq 1$. This concludes our proof. □

Showing Transience

By similar methods, we next prove the transient case.

Proof of Lemma 1 (Transience). We aim to use Lemma 4. We first check that

$$\inf_{n,t} v_{n,t}(x) > 0.$$

Since x is the unique neighbor of s, recall that $v_{n,t}(x) = r_t(s, x)/\mathcal{R}_{n,t}(s, \partial V_n)$ for $n \geq 1$. Also, by the boundedness condition, there exists $C_1 > 0$ such that a.s. $|r_0(e) - r_t(e)| \leq$

$\delta_e \leq C_1$ where $\delta_e := \sum_{t=0}^{\infty} |r_t(e) - r_{t+1}(e)|$. Letting $i := i^{n,0}_{s,\partial V_n}$, Thomson's principle gives

$$\mathcal{R}_{n,t}(s, \partial V_n) \leq \sum_{e \in E_n} i^2(e) r_t(e) \leq \sum_{e \in E_n} i^2(e)(r_0(e) + \delta_e) \leq \mathcal{R}_{n,0}(s, \partial V_n) + C_1.$$

Moreover, as the RWCE is bounded, there exists $C_2 > 0$ such that $w_t(s, x) \leq C_2$ for all $t \in \mathbb{N}$. Hence,

$$v_{n,t}(x) \geq \frac{1/C_2}{\mathcal{R}_{n,0}(s, \partial V_n) + C_1} \geq \frac{1/C_2}{\lim\limits_{n \to \infty} \mathcal{R}_{n,0}(s, \partial V_n) + C_1}$$

since $\mathcal{R}_{n,0}(s, \partial V_n)$ is increasing in n. As (G, w_0) is transient, we conclude that

$$\inf_{n,t} v_{n,t}(x) > 0$$

as desired.

Next, we show that the condition on $\beta_{n,t}$ holds. Note that

$$\beta_{n,t} \geq v_{n,t+1}(x) \geq \inf_{n,t} v_{n,t}(x)$$

for any $n \geq 1$ and $t \in \mathbb{N}$. Moreover, let $S := \{t \in \mathbb{N} : \sigma_t > 1/(2C_2)\}$ where $\sigma_t = \sum_{e \in E} |r_t(e) - r_{t+1}(e)|$. Since $\sum_{t=0}^{\infty} \sigma_t \leq C_1$ a.s., it follows that $|S| \leq 2C_1C_2$ almost surely. Beginning with Corollary 3, we have

$$\prod_{t=0}^{\infty} \beta_{n,t} \geq \prod_{t \in S} \beta_{n,t} \cdot \prod_{t \notin S} (1 - C_2\sigma_t) \geq \prod_{t \in S} \beta_{n,t} \cdot \exp\left(-\sum_{t \notin S} \frac{C_2\sigma_t}{1 - C_2\sigma_t}\right).$$

Since $1/(1 - C_2\sigma_t) \leq 2$ if $t \notin S$, we conclude that a.s.

$$\prod_{t=0}^{\infty} \beta_{n,t} \geq \left(\inf_{n,t} v_{n,t}(x)\right)^{\lceil 2C_1C_2 \rceil} \exp\left(-2C_2 \sum_{t \notin S} \sigma_t\right) \geq \left(\inf_{n,t} v_{n,t}(x)\right)^{\lceil 2C_1C_2 \rceil} e^{-2C_1C_2}.$$

Choosing the final value as b_n in Lemma 3 for all $n \geq 1$, we conclude our proof. $\square$

3.3 The Multiple Neighbor Case

We now consider the general case where the origin s has multiple neighbors. As mentioned in section 2.1, the idea is to attach a new vertex s' to s and construct a new RWCE on the new graph.

3.3.1 Desired Properties

Here, we describe the desired properties of the new RWCE. Recall that s is the origin of G and $\{\langle X_t, w_t\rangle\}_{t\in\mathbb{N}}$ is a bounded RWCE on G that satisfies the boundedness condition. Also, w_0 is deterministic. First attach a vertex s' to s to get $G' = (V', E')$ where $V' = V \cup \{s'\}$ and $E' = E \cup \{s, s'\}$. We aim to construct a new bounded RWCE $\{\langle X'_t, w'_t\rangle\}_{t\in\mathbb{N}}$ on G' whose recurrence (resp. transience) implies the recurrence (resp. transience) of $\{\langle X_t, w_t\rangle\}_{t\in\mathbb{N}}$. Then, viewing s' as the origin of G', we can apply Lemma 1 to $\{\langle X'_t, w'_t\rangle\}_{t\in\mathbb{N}}$ if it also satisfies the boundedness condition.

Note that the restriction of $\{\langle X'_t, w'_t\rangle\}_{t\in\mathbb{N}}$ to G induces a natural RWCE on G. If this induced RWCE is equal in distribution to $\{\langle X_t, w_t\rangle\}_{t\in\mathbb{N}}$, we claim that we have our desired implication of recurrence or transience. More concretely, let N_t be the number of edges in E traversed by $(X'_0, \ldots, X'_t)$ for each $t \in \mathbb{N}$. Also define stopping times $\tau_k = \inf\{t \in \mathbb{N} : N_t = k\}$ for $k \in \mathbb{N}$. Then, we say the *RWCE induced by* $\{\langle X'_t, w'_t\rangle\}_{t\in\mathbb{N}}$ on G is $\{\langle Y_k, \omega_k\}_{k\in\mathbb{N}}$ where $Y_k = X'_{\tau_k}$ and $\omega_k = w'_{\tau_k} \restriction_E$ for each $k \in \mathbb{N}$. In particular, the vertex sequence $\{Y_k\}_{k\in\mathbb{N}}$ simply tracks the edges in E crossed by $\{X'_t\}_{t\in\mathbb{N}}$.

We now explain how the desired implications follow if $\{\langle X'_t, w'_t\rangle\}_{t\in\mathbb{N}}$ is bounded and $\{\langle Y_k, \omega_k\rangle\}_{k\in\mathbb{N}}$ equals $\{\langle X_t, w_t\rangle\}_{t\in\mathbb{N}}$ in distribution. First consider when $\{\langle X'_t, w'_t\rangle\}_{t\in\mathbb{N}}$ is recurrent and thus a.s. visits s' infinitely often. If s is visited finitely often in $\{\langle Y_k, \omega_k\rangle\}_{k\in\mathbb{N}}$, then the only way s' can be visited infinitely often in $\{\langle X'_t, w'_t\rangle\}_{t\in\mathbb{N}}$ is by alternating between s and s' infinitely many times in a row. However, this happens with probability zero as $\{\langle X'_t, w'_t\rangle\}_{t\in\mathbb{N}}$ is bounded and the probability of jumping from s to s' is bounded above by some number less than 1. Hence, s is a.s. visited infinitely often in $\{\langle Y_k, \omega_k\rangle\}_{k\in\mathbb{N}}$ which implies recurrence of $\{\langle X_t, w_t\rangle\}_{t\in\mathbb{N}}$. Next, assume that $\{\langle X'_t, w'_t\rangle\}_{t\in\mathbb{N}}$ is transient and thus a.s visits s finitely often. Since we only remove vertices when obtaining $\{Y_k\}_{k\in\mathbb{N}}$ from $\{X'_t\}_{t\in\mathbb{N}}$, it follows that s is a.s. visited finitely often in $\{\langle Y_k, \omega_k\rangle\}_{k\in\mathbb{N}}$ which implies transience of $\{\langle X_t, w_t\rangle\}_{t\in\mathbb{N}}$.

3.3.2 Formal Construction

Here, we construct our desired $\{\langle X'_t, w'_t\rangle\}_{t\in\mathbb{N}}$. We determine the random variables sequentially, beginning with X'_0, then w'_0, then X'_1, then w'_1, and so on. The key idea is to determine w'_t as if we were determining w_{N_t} given $(Y_0, Y_1, \ldots, Y_{N_t}, \omega_0, \ldots, \omega_{N_t-1})$ as the history. If $\{X'_{t-1}, X'_t\} = \{s, s'\}$, however, then $N_t = N_{t-1}$ and in this case we freeze the weights by letting $w'_t = w'_{t-1}$. Indeed, we unfreeze afterwards as soon as an edge in E is crossed.

For notational simplicity, let $\mathcal{W}_E := (0,\infty)^E$ and $w'_{t,E} := w'_t \restriction_E$ for any $w'_t \in \mathcal{W}^{E'}$. We now give the measure-theoretic construction of $\{\langle X'_t, w'_t\rangle\}_{t\in\mathbb{N}}$. Let $X'_0 = X_0$ in distribution and let $w'_0 \restriction_E = w_0$. Also let $w'_t(s, s') = 1$ for all $t \in \mathbb{N}$. Then, it remains to define the conditional probabilities $\mathbb{P}(w'_{t+1,E} \in A \mid \mathcal{G}'_t)$ for each $t \in \mathbb{N}$ and $A \in \mathcal{B}(\mathcal{W}_E)$ where $\mathcal{G}'_t := \sigma(X'_0, \ldots, X'_{t+1}, w'_0, \ldots, w'_t)$. In this process, we will involve standard notions from probability theory such as regular conditional probabililty distributions (RCPD), Polish spaces and the Doob-Dynkin functional representation, and extension theorems (Kolmogorov, Carathéodory). We include [3, 7, 11] as a reference for these notions.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ denote the underlying probability space and $\widehat{\mathbf{P}}_{w_{t+1}|\mathcal{G}_t}(\cdot,\cdot) : \mathcal{B}(\mathcal{W}_E) \times \Omega \to [0,1]$ denote the RCPD of w_{t+1} given $\mathcal{G}_t := \sigma(X_0, \ldots, X_{t+1}, w_0, \ldots, w_t)$ for $t \in \mathbb{N}$. The RCPD exists since E is countable and $\mathcal{W}_E$ is a Polish space. By the Doob-Dynkin functional representation, we have

$$\widehat{\mathbf{P}}_{w_{t+1}|\mathcal{G}_t}(A, \omega) = f_{t,A}(X_0(\omega), \ldots, X_{t+1}(\omega), w_0(\omega), \ldots, w_t(\omega))$$

for any $A \in \mathcal{B}(W_E)$ and $\omega \in \Omega$ where $f_{t,A} : V^{t+2} \times (\mathcal{W}_E)^{t+1} \to [0,1]$ is some measurable function. Moreover, we know that $f_{t,\cdot}(X_0, \ldots, X_{t+1}, w_0, \ldots, w_t)$ is a.s. a probability measure on $(\mathcal{W}_E, \mathcal{B}(\mathcal{W}_E))$.

To conclude, when $N_{t+1} = N_t + 1$, we require

$$\mathbb{P}(w'_{t+1,E} \in A \mid \mathcal{G}'_t) = f_{N_t,A}(Y_0, \ldots, Y_{N_t+1}, \omega_0, \ldots, \omega_{N_t}). \tag{3.2}$$

Otherwise, if $N_{t+1} = N_t$, then given $\mathcal{G}'_t$ we require $w'_{t+1} = w'_t$. Note that if we know

$$(X_0, \ldots, X_t, w_0, \ldots, w_{t-1}) \stackrel{d}{=} (Y_0, \ldots, Y_t, \omega_0, \ldots, \omega_{t-1})$$

for $t \leq k+1$, then (3.2) is a well-defined probability for $t \leq \tau_{k+1} - 1$. We will show this equality in distribution while proving $\{\langle X_t, w_t\rangle\}_{t\in\mathbb{N}} \stackrel{d}{=} \{\langle Y_t, \omega_t\rangle\}_{t\in\mathbb{N}}$ in the following section.

3.3.3 Verifying the Construction

Here, we justify our construction through the following lemma.

Lemma 6. *We have $\{\langle X_t, w_t\rangle\}_{t\in\mathbb{N}} \stackrel{d}{=} \{\langle Y_t, \omega_t\rangle\}_{t\in\mathbb{N}}$. In particular,* (3.2) *is well-defined for all $t \in \mathbb{N}$.*

Proof. By Kolmogorov's extension theorem for Polish spaces, it suffices to show that

$$(X_0, w_0, \ldots, X_{t-1}, w_{t-1}, X_t) \stackrel{d}{=} (Y_0, \omega_0, \ldots, Y_{t-1}, \omega_{t-1}, Y_t), \tag{3.3}$$
$$(X_0, w_0, \ldots, X_{t-1}, w_{t-1}, X_t, w_t) \stackrel{d}{=} (Y_0, \omega_0, \ldots, Y_{t-1}, \omega_{t-1}, Y_t, \omega_t) \tag{3.4}$$

for each $t \in \mathbb{N}$. We proceed by strong induction on $t \in \mathbb{N}$. If $t = 0$, both claims follow since $\tau_0 = 0$. Next, assume that both claims hold for all $t \leq k$ where $k \in \mathbb{N}$. We first show that (3) also holds for $t = k+1$. By Carathéodory's extension theorem, it suffices to show that

$$\mathbb{P}\left(\{X_i = x_i\}_{i=0}^{k+1}, \{w_j \in E_j\}_{j=0}^{k}\right) = \mathbb{P}\left(\{Y_i = x_i\}_{i=0}^{k+1}, \{\omega_j \in E_j\}_{j=0}^{k}\right)$$

for any $x_0, \ldots, x_{k+1} \in V$ and $E_0, \ldots, E_t \in \mathcal{B}(\mathcal{W}_E)$. Beginning with the right-hand side, we have

$$\begin{aligned}\mathbb{P}\left(\{Y_i = x_i\}_{i=0}^{k+1}, \{\omega_j \in E_j\}_{j=0}^{k}\right) &= \mathbb{E}\left[\prod_{i=0}^{k+1} 1_{Y_i = x_i} \prod_{j=0}^{k} 1_{\omega_j \in E_j}\right] \\ &= \mathbb{E}\left[\prod_{i=0}^{k} 1_{Y_i = x_i} \prod_{j=0}^{k} 1_{\omega_j \in E_j} \mathbb{E}\left[1_{Y_{k+1} = x_{k+1}} \mid \mathcal{F}'_{\tau_k}\right]\right]\end{aligned}$$

where $\mathcal{F}'_k = \sigma(X'_0, \ldots, X'_t, w'_0, \ldots, w'_k)$. If $Y_k \neq s$, then we have $\mathbb{E}\left[1_{Y_{k+1} = x_{k+1}} \mid \mathcal{F}'_{\tau_k}\right] = \omega_k(Y_k, x_{k+1})/\omega_k(Y_k)$. Otherwise, if $Y_k = s$, then we have

$$\mathbb{E}\left[1_{Y_{k+1} = x_{k+1}} \mid \mathcal{F}'_{\tau_k}\right] = \sum_{k=0}^{\infty} \frac{1}{(\omega_k(s) + 1)^k} \cdot \frac{\omega_k(s, x_{k+1})}{\omega_k(s) + 1} = \frac{\omega_k(s, x_{k+1})}{\omega_k(s)}.$$

Hence, we can write

$$\mathbb{P}\left(\{Y_i = x_i\}_{i=0}^{k+1}, \{\omega_j \in E_j\}_{j=0}^{k}\right) = \mathbb{E}[g_k(Y_0, \ldots, Y_k, \omega_0, \ldots, \omega_k)]$$

where

$$g_k(Y_0, \ldots, Y_k, \omega_0, \ldots, \omega_k) = \frac{\omega_k(Y_k, x_{k+1})}{\omega_k(Y_k)} \cdot \prod_{i=0}^{k} 1_{Y_i = x_i} \prod_{j=0}^{k} 1_{\omega_j \in E_j}.$$

By the inductive hypothesis, we have

$$\mathbb{E}[g_k(Y_0, \ldots, Y_k, \omega_0, \ldots, \omega_k)] = \mathbb{E}[g_k(X_0, \ldots, X_k, w_0, \ldots, w_k)].$$

Working backwards, we see that

$$\mathbb{E}[g_k(X_0, \ldots, X_k, w_0, \ldots, w_k)] = \mathbb{P}\left(\{X_i = x_i\}_{i=0}^{k+1}, \{w_j \in E_j\}_{j=0}^{k}\right).$$

This gives (3) for $t \leq k+1$. It follows that (3.2) is well-defined for $t \leq \tau_{k+1} - 1$.

Next, we show that (4) also holds for $t = k+1$. Again, it suffices to show that

$$\mathbb{P}\left(\{X_i = x_i\}_{i=0}^{k+1}, \{w_j \in E_j\}_{j=0}^{k+1}\right) = \mathbb{P}\left(\{Y_i = x_i\}_{i=0}^{k+1}, \{\omega_j \in E_j\}_{j=0}^{k+1}\right)$$

for any $x_0, \ldots, x_{k+1} \in V$ and $E_0, \ldots, E_{k+1} \in \mathcal{B}(\mathcal{W}_E)$. Beginning with the right-hand side, we have

$$\begin{aligned}
\mathbb{P}\left(\{Y_i = x_i\}_{i=0}^{k+1}, \{\omega_j \in E_j\}_{j=0}^{k+1}\right) &= \mathbb{E}\left[\prod_{i=0}^{k+1} 1_{Y_i = x_i} \prod_{j=0}^{k+1} 1_{\omega_j \in E_j}\right] \\
&= \mathbb{E}\left[\prod_{i=0}^{k+1} 1_{Y_i = x_i} \prod_{j=0}^{k} 1_{\omega_j \in E_j} \mathbb{E}\left[1_{\omega_{k+1} \in A_{k+1}} \mid \mathcal{G}'_{\tau_{k+1}-1}\right]\right].
\end{aligned}$$

Since (3.2) is well-defined for $t \leq \tau_{k+1} - 1$, we have

$$\mathbb{E}[1_{\omega_{k+1} \in E_{k+1}} \mid \mathcal{G}'_{\tau_{k+1}-1}] = f_{k, E_{k+1}}(Y_0, \ldots, Y_{k+1}, \omega_0, \ldots, \omega_k).$$

Hence, we can write

$$\mathbb{P}\left(\{Y_i = x_i\}_{i=0}^{k+1}, \{\omega_j \in E_j\}_{j=0}^{k+1}\right) = \mathbb{E}[h_k(Y_0, \ldots, Y_{k+1}, \omega_0, \ldots, \omega_k)]$$

where

$$h_k(Y_0, \ldots, Y_{k+1}, \omega_0, \ldots, \omega_t) = f_{k,E_{k+1}}(Y_0, \ldots, Y_{k+1}, \omega_0, \ldots, \omega_k) \cdot \prod_{i=0}^{k+1} 1_{Y_i = x_i} \prod_{j=0}^{k} 1_{\omega_j \in E_j}.$$

Since (3) holds for $t = k + 1$, we have

$$\mathbb{E}[h_k(Y_0, \ldots, Y_{k+1}, \omega_0, \ldots, \omega_k)] = \mathbb{E}[h_k(X_0, \ldots, X_{k+1}, w_0, \ldots, w_k)].$$

Working backwards, we see that

$$\mathbb{E}[h_k(X_0, \ldots, X_{k+1}, w_0, \ldots, w_k)] = \mathbb{P}\left(\{X_i = x_i\}_{i=0}^{k+1}, \{w_j \in E_j\}_{j=0}^{k+1}\right).$$

This gives (4) for $t \leq k + 1$. By induction, we conclude our proof. $\square$

3.3.4 Proof of Main Result

We are now ready to prove Theorem 1 in full generality.

Proof of Theorem 1. We aim to use Lemma 1. We will check the necessary conditions for G' and $\{\langle X'_t, w'_t \rangle\}_{t \in \mathbb{N}}$ constructed above. First, choose s' as the origin of G'. Then, s' has a single neighbor s and w'_0 is deterministic. Since $w_t(s, s') = 1$ for all $t \in \mathbb{N}$, combining this with Lemma 6 it follows that $\{\langle X'_t, w'_t \rangle\}_{t \in \mathbb{N}}$ is bounded. Finally, since the weights are frozen when $\{\langle X'_t, w'_t \rangle\}_{t \in \mathbb{N}}$ traverses along $\{s, s'\}$, it follows that

$$\sum_{t,e} |r'_t(e) - r'_{t+1}(e)| \overset{a.s.}{=} \sum_{k,e} |r'_{\tau_k}(e) - r'_{\tau_{k+1}}(e)| \overset{d}{=} \sum_{k,e} |r_k(e) - r_{k+1}(e)|$$

where the second equality is by Lemma 6. Hence, $\sum_{t,e} |r'_t(e) - r'_{t+1}(e)|$ is also bounded. To conclude, by Lemma 1, we see that $\{\langle X'_t, w'_t \rangle\}_{t \in \mathbb{N}}$ inherits the recurrence or transience of (G', w'_0). Moreover, note that

$$\mathcal{R}'(s', \partial V'_{n+1}) = 1 + \mathcal{R}(s, \partial V_n)$$

where $\mathcal{R}'$ is the effective resistance function on (G', w'_0) with s' as the origin and $\mathcal{R}$ is the effective resistance function on (G, w_0) with s as the origin. Hence, it follows that (G, w_0) and (G', w'_0) are either both recurrent or both transient. To conclude, if

(G, w_0) is recurrent, it follows that (G', w_0'), then $\{\langle X_t', w_t' \rangle\}_{t\in\mathbb{N}}$, then $\{\langle X_t, w_t \rangle\}_{t\in\mathbb{N}}$ are also recurrent where the last implication was discussed in section 4.1. The case is the same for transience and we conclude our proof. $\square$

Chapter 4

Concluding Remarks

We conclude by discussing insight from our attempts and results along with possible future work.

First, we discuss the martingale method for showing recurrence or transience. In order to tackle the monotone-bounded problem using this method, we remark that one must construct a super/submartingale that only works when the RWCE is nonadaptive. For instance, one could construct a supermartingale that uses information about all weights $\{w_t\}_{t=0}^{\infty}$ in advance, which would not be well-defined if the RWCE is adaptive. Moreover, it could also be possible that the constructed function only has the martingale property when the RWCE is nonadaptive. In either case, we remark that one must crucially exploit the fact that the RWCE is nonadaptive when constructing the supermartingale. Of course, this was not exploited in our construction (section 3.2.1) or in the construction of Amir et al. in [2].

Moreover, it would be nice if one could improve the double summation in our main result by changing the summation over $e \in E$ into a supremum over $e \in E$. One place in our proof that can be greatly improved is our usage of Lemma 5. Here, we used the very rough estimate of bounding the unit flows by one, but in theory one could aim to bound the voltage differences much more accurately. This may help in improving upon our main result.

Finally, we discuss the case of RWCEs with deterministic weights uniformly converging to a recurrent graph. In this direction, it would be interesting to improve upon Corollary 2 by investigating the slowest rate of convergence that is still able to guarantee recurrence: Recall that in Corollary 2, we showed $\sum_{t\in\mathbb{N}} \varepsilon_t < \infty$ is sufficient, but ε_t

that decay even slower should also be able to guarantee recurrence on general graphs. Moreover, it would be nice to remove the assumption on bounded degree that we make in Corollary 2.

The final problem in this direction is determining the location of the phase-transition for the outward-biased RWCE on trees we discussed at the end of section 2.2. Given biases of $\varepsilon_t = 1/t^\alpha$, it was shown in [14] that $\alpha = 1/2$ is the threshold for $G = \mathbb{N}$. In a sense, note that $\mathbb{N}$ is the "least" transient infinite graph since there is only one path to infinity. Even in this graph, any $\alpha < 1/2$ creates a bias too strong that makes the RWCE transient. Considering general recurrent trees, these are now more "transient" than $\mathbb{N}$ since there are more paths to infinity. Hence, it is not obvious if the threshold for recurrence will also be at $\alpha = 1/2$, or whether it would be at a value strictly larger than 1/2. Answering this for general trees would require a completely different approach compared to our usage of the central limit theorem in section 2.2.

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